FLOW OF A RAREFIED GAS IN A LONG FINITE CIRCULAR CAPILLARY

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Poiseuille flow in an infinitely long circular capillary was studied numerically in [1] and by variational methods in [2, 3]. Papers on this problem appear frequently in proceedings of conferences on the dynamics of a rarefied gas (e.g. [4]). Actually the length of a channel is finite. In this case the special problems of free-molecule [5] and viscous [6] flows of a gas are generally treated. It is therefore of interest to solve the problem over the whole range of Knudsen numbers $\mathrm{Kn} = \lambda_1/r$, where λ_1 is the molecular mean free path, and r is the radius of the capillary. The problem is simplified somewhat by assuming that the channel is long enough so that the ratio of its length to its radius $L_1 = l/r \gg 1$. We choose the Bhatnagar–Gross–Krook (BGK) equation [7] as the input equation for the distribution function for molecular velocities. We solve the linearized problem by reducing the input equation to a Fredholm integral equation for the mean molecular speed, which we solve by the Galerkin method [8, 9].

1. Suppose gas flows from the first reservoir into the second. The temperatures of the gas are the same in both reservoirs, and the number densities are negligibly different. The z axis is oriented along the direction of gas flow, as shown in Fig. 1. The x and y axes lie in the plane of a middle cross section of the channel, and the origin is at the center of the capillary.

We take the following scales:

$$r,\ n_1,\ h^{1/2} = (2RT_1)^{1/2},\ T_1,\ n_1 h^{-3/2},\ \eta_1 = n_1 m v \lambda_1/2$$

for length, number density of the gas, mean velocity of the gas **u** and the speed of a molecule **c**, temperature T, the distribution function f for molecular velocities, and the viscosity η respectively. Here R is the gas constant, $v = (8RT_1/\pi)^{1/2}$; $\lambda_1 = (\sqrt{2}\pi n_1 \sigma^2)^{-1}$; and m and σ are the mass and diameter of a molecule. The subscripts 1 and 2 denote parameters of the gas in the first and second reservoirs.

Molecules from the first reservoir enter the channel through the cross section $z = -L$ $(L = L_1/2)$ with a distribution function f_1, and from the second reservoir through the cross section $z = L$ with a distribution function f_2. We assume that particles are diffusely reflected from the channel walls with a distribution function f_3. We assume that the gas density is a linear function of z:

$$f_1 = \pi^{-3/2} \exp(-c^2),\ f_2 = n_2 f_1,\ f_3 = [1 - K(L+z)] f_1,\quad K = (1 - n_2)/L_1. \tag{1.1}$$

Equations (1.1) are boundary conditions for the BGK equation [10]. They determine the distribution functions for molecules whose velocities are directed into the capillary:

$$\mathbf{c}\,\partial f/\partial \mathbf{s} = \delta n (f_0 - f),\ f_0 = f_1(1 + \nu + 2\mathbf{cu}),\quad n = 1 + \nu,\ \nu = -K(L+z),\ \delta = \sqrt{\pi}/2\mathrm{Kn}. \tag{1.2}$$

Here f_0 is the linearized local Maxwellian distribution function for the molecules, n is the number density of the gas molecules, **s** is the radius vector of the observation point with the direction Ω (Ω is a unit vector in the direction of the velocity $\mathbf{c} = c\Omega$). We assume that the mean speed of the gas $u \ll 1$, and that the absolute magnitude of the perturbation of the number density $|\nu| \ll 1$. We write the required distribution function in the form $f = f_1(1 + \nu + h_1)$, where the perturbation of the distribution function $|h_1| \ll 1$.

We assume that the channel is long enough so that end effects can be neglected, and that **u** has only a z component. Then the velocity will not depend on z, and all the cross sections

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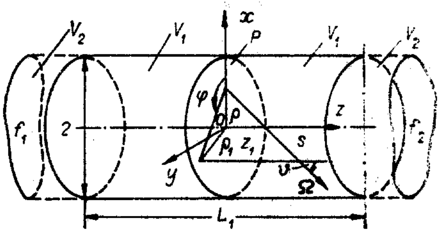

Fig. 1

of the capillary become equivalent. Therefore, we can choose the cross section $z=0$ for study. In this formulation of the problem the mean speed of the gas depends only on the distance ρ from the observation point to the z axis. Using the definition of the mean speed of the gas in terms of the distribution function

$$u = \int fc\cos\vartheta d\mathbf{c}, \quad d\mathbf{c} = c^2 dc d\Omega, \quad d\Omega = \sin\vartheta d\vartheta d\varphi$$

and Eq. (1.2), we obtain an integral equation for the mean speed of molecules in the z direction:

$$u = \pi^{-3/2}\int\limits_{V_1}(K + 2\delta u) I_3(\delta s)\cos^2\vartheta ds d\Omega, \tag{1.3}$$

where V_1 is the volume of the channel, ϑ is the angle between the Ω vector and the z axis, s is the length of the radius vector $\mathbf{s}$, and φ is the polar angle in the $z=0$ plane (Fig. 1). The Abramowitz integral I_3 [11, 12] appears in the Fredholm integral equation of the second kind (1.3). In the general case this integral is defined as

$$I_m(t) = \int\limits_0^\infty c^m \exp(-c^2 - t/c)\, dc.$$

2. We solve Eq. (1.3) by the Galerkin method [8]. We choose 1 and ρ^2, where $\rho^2 = x^2 + y^2$, as basis functions. The expansion of the mean speed in the system of basis functions has the form

$$u/K = C - D\rho^2, \tag{2.1}$$

where the expansion coefficients C and D depend on the parameters δ and L. Substituting Eq. (2.1) into (1.3) and integrating over the middle cross section P (Fig. 1) with the weights 1 and ρ^2, we obtain a system of linear algebraic equations for the unknown coefficients C and D:

$$b_{11}C + b_{12}D = b_1, \; b_{21}C + b_{22}D = b_2. \tag{2.2}$$

The coefficients b_{ij} and b_i contain the multiple integrals

$$b_{ij} = \frac{(-1)^{i+1}}{2(i+j-1)} + c_{ij}, \quad b_i = \frac{c_{i1}}{2\delta}, \quad i, j = 1, 2,$$
$$c_{ij} = (-1)^{j+1}\frac{8\delta}{\pi^{3/2}}\int\limits_0^1 \rho^{2i-1}d\rho \int\limits_{V_1/4} I_3(\delta s)\rho_1^{2(j-1)}\cos^2\vartheta ds d\Omega, \tag{2.3}$$

where $\rho_1^2 = \rho^2\sin^2\vartheta + \rho^2 - 2\rho(\rho - \rho_0\cos\varphi_0)s/s_0$ (the subscript 0 refers to a point on the surface of the volume V_1 of the capillary). As a result of the symmetry with respect to z and φ we integrate over one quarter of the volume. Knowing the values of C and D, we find the reduced flow rate of the gas:

$$Q = \frac{4}{K}\int\limits_0^1 \rho u d\rho = 2C - D. \tag{2.4}$$

Figure 2 illustrates the results of calculations with Eq. (2.4). Curves 1-4 correspond to channel lengths $L_1 = \infty$, 40, 20, 10. The computer time is considerably shortened by using the expansion of I_3 (δs) in powers of δs when $\delta L \leqslant 2$ [11, 12].

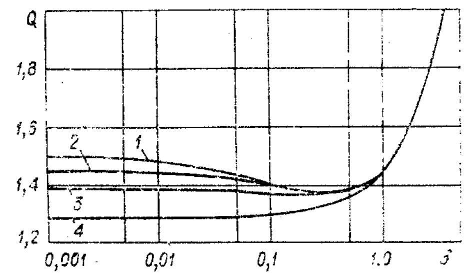

Fig. 2

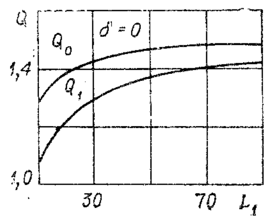

Fig. 3

For $\delta L > 2$ it is convenient to use the asymptotic representations of the Abramowitz integrals. With this in mind we separate out of each of the coefficients b_{ij} and b_i the part giving the coefficients a_{ij} for the equations determining the solution, i.e. the coefficients C and D for an infinitely long channel:

$$a_{ij} = \frac{(-1)^{i+1}}{2(i+j-1)} + J_{ij}, \quad a_i = -\frac{1}{2\delta} J_{1i}. \tag{2.5}$$

The symbols J_{ij} denote the definite integrals

$$J_{ij} = (-1)^i \frac{2\delta}{\pi} \int_0^1 \rho^{2j-1} d\rho \int_{P/2} \rho_1^{2(i-1)} I_0(\delta s)\, ds\, d\varphi,$$

where $\rho_1^2 = \rho^2 + s^2 + 2\rho s \cos\varphi$. As a result of the symmetry with respect to φ we integrate over half the cross section P of the channel (Fig. 1). Then we can write the coefficients in Eqs. (2.2) in the form

$$b_{ij} = a_{ij} + c'_{ij}, \quad b_i = a_i - c'_{i1}/(2\delta).$$

The quantities c'_{ij} contain integrals over the volume V_2 which extends the length of the channel under consideration to infinity. The formula for c'_{ij} is obtained from the relation for c_{ij} in (2.3) by replacing V_1 by V_2.

3. It is of interest to investigate the behavior of the reduced flow rate Q (2.4) for very large and very small values of the rarefaction parameter δ. As $\delta \to 0$, which corresponds to free-molecule flow of the gas, we obtain

$$Q = Q_0 - \delta \ln L_1 + 3.044\delta - 3.395\delta/L \ (\delta L \ll 1).$$

Here Q_0 is the flow rate of the gas through a channel of length L_1 in free-molecule flow:

$$Q_0 = [L^3 - \sqrt{(4+L^2)^3} + 6L + 8]/(3\sqrt{\pi}).$$

This result differs from the more accurate formula [5]

$$Q_1 = \frac{2L_1}{\sqrt{\pi}} \left\{ 1 + L^2 - L\sqrt{1+L^2} - \frac{2}{9} \frac{[L^3 - 2 + (2-L^2)\sqrt{1+L^2}]^2}{L\sqrt{1+L^2} - \operatorname{arsh} L} \right\}.$$

The error is 19% for $L_1 = 10$, and decreases to zero with increasing channel length (Fig. 3). This inaccuracy can be eliminated if Eq. (1.3) is supplemented by the condition that there is no flow of gas at the wall which, for free-molecule flow, is transformed into the Clausing equation [13].

When $\delta L \gg 1$, it is easy to derive formulas containing the well-known expressions

$$Q = \frac{8}{3\sqrt{\pi}} + \delta \ln \delta - \frac{\delta}{\sqrt{3t}} \exp(-3t)\,(\delta L \gg 1, \ \delta \ll 1); \tag{3.1}$$

$$Q = \frac{\delta}{4} + \frac{4+\pi}{4\sqrt{\pi}} - \frac{\delta^3}{\sqrt{3}L^2} t^2 \exp(-3t)\,(\delta L \gg 1, \ \delta \gg 1), \tag{3.2}$$

where $t = (\delta L/2)^{2/3}$. The first terms in Eqs. (3.1) and (3.2) correspond to values of the rate of flow of gas through an infinitely long channel under free-molecule and viscous flow conditions respectively [14]. The following terms decrease very rapidly with increasing δL.

The solution Q which follows from Eqs. (2.5) for an infinitely long channel (Table 1) is in good agreement with data in [1, 2] on Q_2 and Q_3 obtained by numerical and variational methods respectively.

TABLE 1

δ	0	0,01	0.1	1,0	2,0	3,0	4,0	5,0
Q	1,5045	1,4763	1,4039	1,4576	1,6559	1,8772	2,1079	2,3438
Q_2	1,5045	1,4768	1,4043	1,4594	1,6608	1,8850	2,1188	2,3578
Q_3	1,5045	1,4801	1,4039	1,4576	1,6559	1,8772	2,1079	2,3438

The results of the solution can be refined by taking end effects into account [6].

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